DE MOIVRE'S THEOREM AND COMPLEX ROOTS

- When solving polynomial equations of degree 'n', we know that we will have exactly $'n'$ roots – i.e. a quadratic has 2 roots, a cubic has 3 roots etc.
- When finding the roots of a complex number, i.e. solving $z^n = a$ (which incidentally is the same as finding the nth root of a: $\sqrt[n]{a}$), again we also expect $'n'$ roots.
- Complex roots have **rotational symmetry**, so they are spread about the origin at equal distances from it and at equal angles from each other.
- When using De Moivre's Theorem to solve equations and calculate roots, we need to set up a process that yields *all* the answers. To do this, we write the complex number not simply as $r\ cis\ \theta$, but as $r\ cis\ (\theta + 2n\pi)$ or $r\ cis\ (\theta + 360°n)$ since adding on multiples of 360° will generate extra solutions.

Example 1: Solve $z^3 = 27\ cis\ (\pi)$.

$$z^3 = 27\ cis\ (\pi + 2n\pi)$$

$$z = \sqrt[3]{27\ cis\ (\pi + 2n\pi)}$$

$$z = (27\ cis\ (\pi + 2n\pi))^{1/3}$$

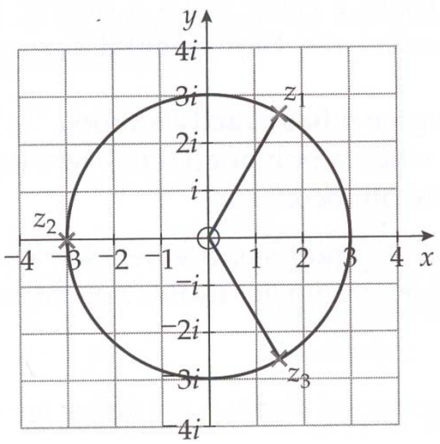
$$z = 27^{1/3}\ cis\ (\tfrac{\pi + 2n\pi}{3})$$ *using De Moivre's Theorem*

$$z = 3\ cis\ (\tfrac{\pi}{3} + \tfrac{2n\pi}{3})$$ *think of this as your general soln, sub in different values of 'n' to generate multiple solutions.*

$$n = 0: z_1 = 3\ cis\left(\frac{\pi}{3} + \frac{2(0)\pi}{3}\right) = 3\ cis\ (\tfrac{\pi}{3})$$

$$n = 1: z_2 = 3\ cis\left(\frac{\pi}{3} + \frac{2(1)\pi}{3}\right) = 3\ cis\ (\pi)$$

$$n = 2: z_3 = 3\ cis\left(\frac{\pi}{3} + \frac{2(2)\pi}{3}\right) = 3\ cis\left(\frac{5\pi}{3}\right) = 3\ cis\ (\tfrac{-\pi}{3})$$



When plotted on an Argand diagram, all three roots are equally spaced around the origin, and each the same distance from the origin.

Example 2: Solve $z^3 = -5 + 13i$.

First, convert $-5 + 13i$ to polar form: $-5 + 13i = 13.93\ cis\ (111°)$

$z^3 = 13.93\ cis\ (111° + 360°n)$

$z = \sqrt[3]{13.93\ cis\ (111° + 360°n)}$

$\mathrm{z} = (13.93\ cis\ (111° + 360°n)^{1/3}$

$z = 13.93^{1/3}\ cis\ (\frac{111°+360°n}{3})$ *using De Moivre's Theorem*

$z = 3\ cis\ (37° + 120°n)$ *think of this as your general soln, sub in different values of 'n' to generate multiple solutions.*

$n = 0$: $z_1 = 2.41\ cis\ (37° + 120°(0)) = 3\ cis\ (37°)$

$n = 1$: $z_2 = 2.41\ cis\ (37° + 120°(1)) = 3\ cis\ (157°)$

$n = 2$: $z_3 = 2.41\ cis\ (37° + 120°(2)) = 3\ cis\ (277°) = 3\ cis\ (-83°)$

In polar form, the three roots are: $1.922 + 1.448i, -2.215 + 0.940i$, and $0.293 - 2.388i$.

Example 3: Calculate the fifth roots of 1024, giving the answers in polar form, and show them on an Argand diagram.

First, convert 1024 to polar form. The point 1024 lies on the real axis on an Argand diagram (since the number is purely real and has no imaginary part) and hence makes an angle of $0°$ with the real axis. So $1024 = 1024\ cis\ (0°)$.

$$z^5 = 1024\ cis\ (0° + 360°n)$$

$$z = \sqrt[5]{1024\ cis\ (360°n)}$$

$$\mathrm{z} = (1024\ cis\ (360°n)^{1/5}$$

$$z = 1024^{1/5}\ cis\ (\frac{360°n}{5})$$ *using De Moivre's Theorem*

$$z = 4\ cis\ (72°n)$$ *think of this as your general soln, sub in different values of 'n' to generate multiple solutions.*

$$n = 0: z_1 = 4\ cis\ \left(72°(0)\right) = 4\ cis\ (0°) = 4$$

$$n = 1: z_2 = 4\ cis\ \left(72°(1)\right) = 4\ cis\ (72°)$$

$$n = 2: z_3 = 4\ cis\ \left(72°(2)\right) = 4\ cis\ (144°)$$

$$n = 3: z_4 = 4\ cis\ \left(72°(3)\right) = 4\ cis\ (216°) = 4\ cis\ (-144°)$$

$$n = 4: z_5 = 4\ cis\ \left(72°(4)\right) = 4\ cis\ (288°) = 4\ cis\ (-72°)$$

In polar form, the five roots are: $4\ cis\ (0°)$, $4\ cis\ (72°), 4\ cis\ (144°)$, $4\ cis\ (-144°)$ and $4\ cis\ (-72°)$.

Delta Ex 32.4 pg 299 Q 3, 9, 10, 11. Extension: the rest